

FRACTALS AND THE SIERPINSKI TRIANGLE

Penelope Allen-Baltera

Dave Marieni

Richard Oliveira

Louis Sievers

Hingham High School

Marlborough High School

Ludlow High School

Trinity High School

The purpose of this project is to explore Sierpinski triangles and the math behind them.

They provide an introduction to fractals, and connections to computer graphics and animation.

The activities also give practice in some geometric techniques and some tie-ins to geometric theorems.

Applicable to entry-level and honors-level Geometry classes.
Possible extensions to higher level courses.

Supplies Needed

Part I: pencil, paper, ruler

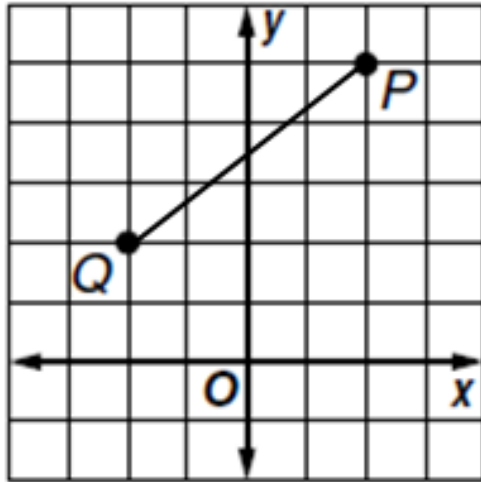
Part II: graph paper, pencil, ruler, colored pencils/highlighters

Part III: computer, chromebook, iPad, other

Optional: overhead projector transparencies, calculator

Warm-up Activity

Find the coordinates of the midpoint of segment PQ.

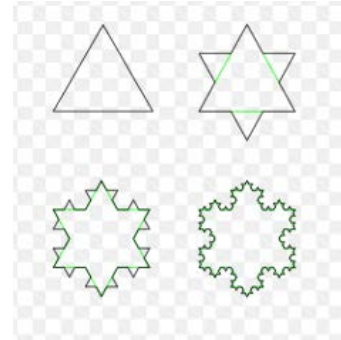
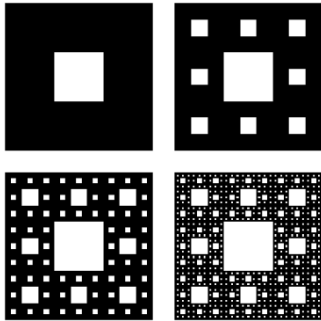


Midpoint Formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

What is a Fractal

- A fractal is a geometric figure in which each part has the same statistical character as the whole. It is a never ending pattern.
- They are created by repeating a simple process over and over again.
- They are useful in modeling structures in which similar patterns recur at progressively smaller scales, and in describing phenomena like crystal growth, fluid turbulence, and galaxy formation.
- Fractals also have applications in computer graphics and animation.



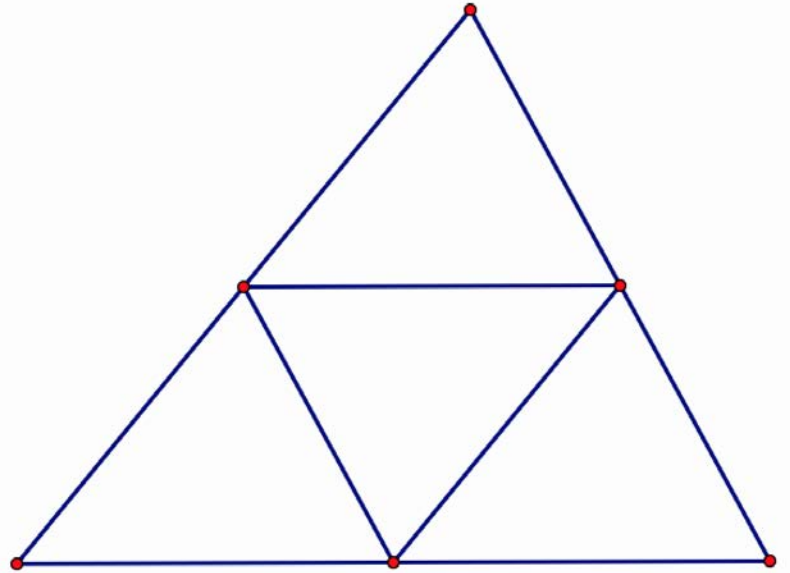
Part I - Make a Sierpinski Triangle

Supplies: paper, ruler, pencil

With a ruler, draw a triangle to cover as much of the paper as possible.

The triangle may be any type of triangle, but it will be easier if it is roughly equilateral.

With pencil and ruler, find the midpoints of each side of the triangle and connect the points.



Additional Steps to Draw a Sierpinski Triangle

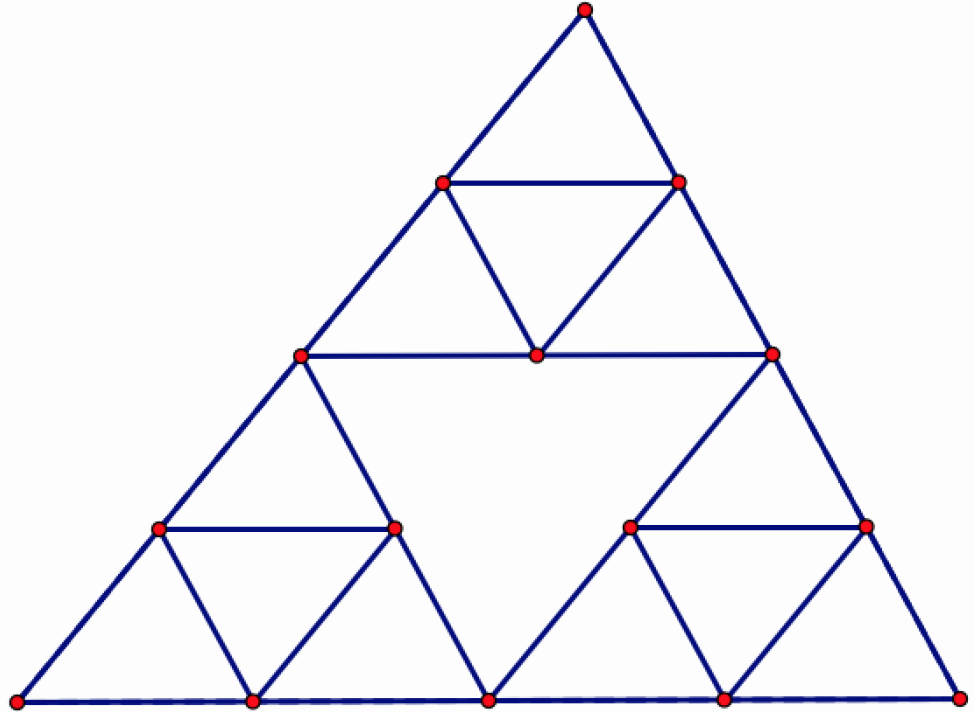
Now repeat the process with the three “outside” triangles of the figure. Find the midpoints of the sides of each of the smaller triangles.

Leaving the center triangle blank, connect the midpoints.

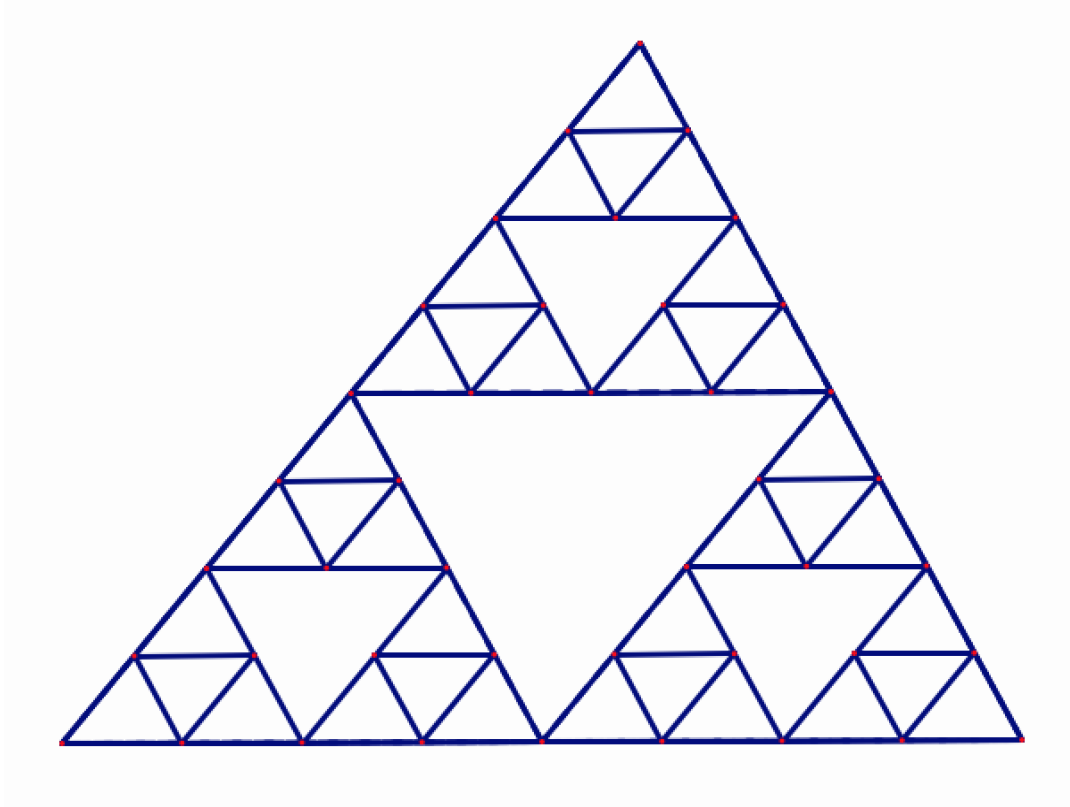
2nd Iteration to Draw a Sierpinski Triangle

Repeat at least one more time, more if possible.

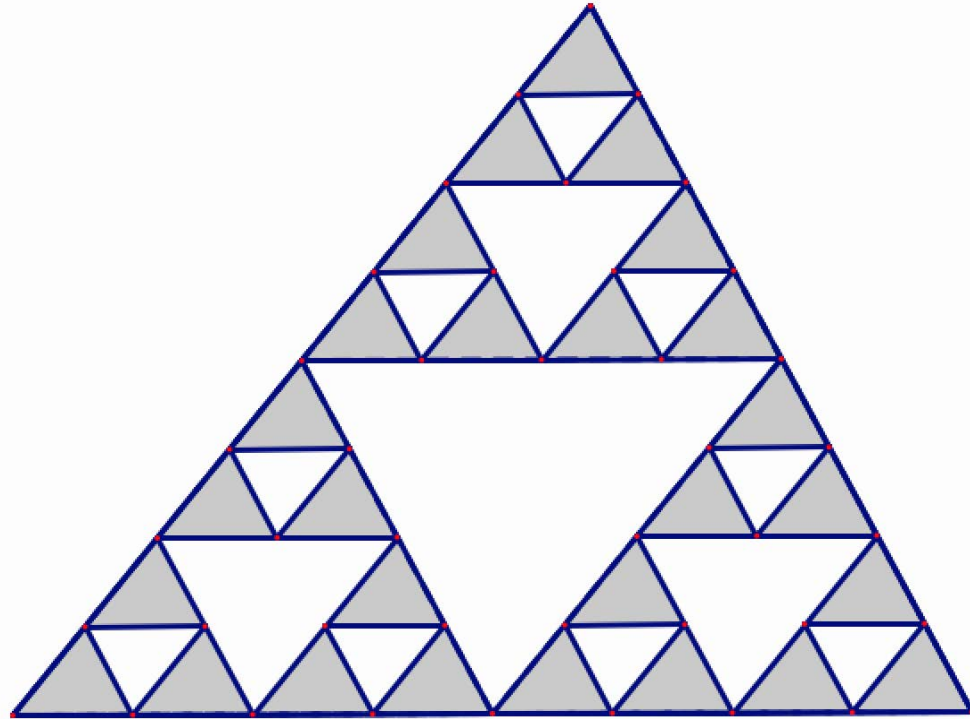
Always leave the middle of each triangle open.



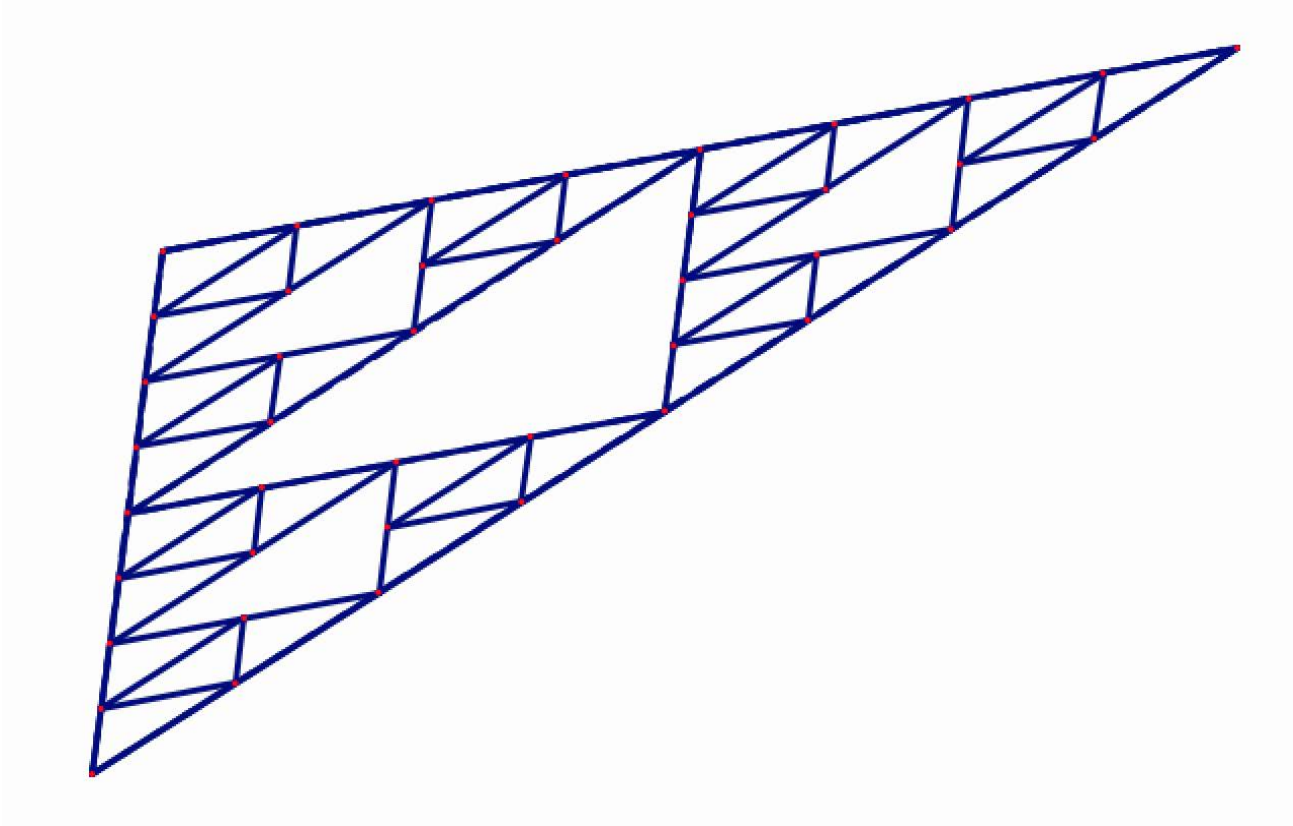
3rd Iteration to Draw a Sierpinski Triangle



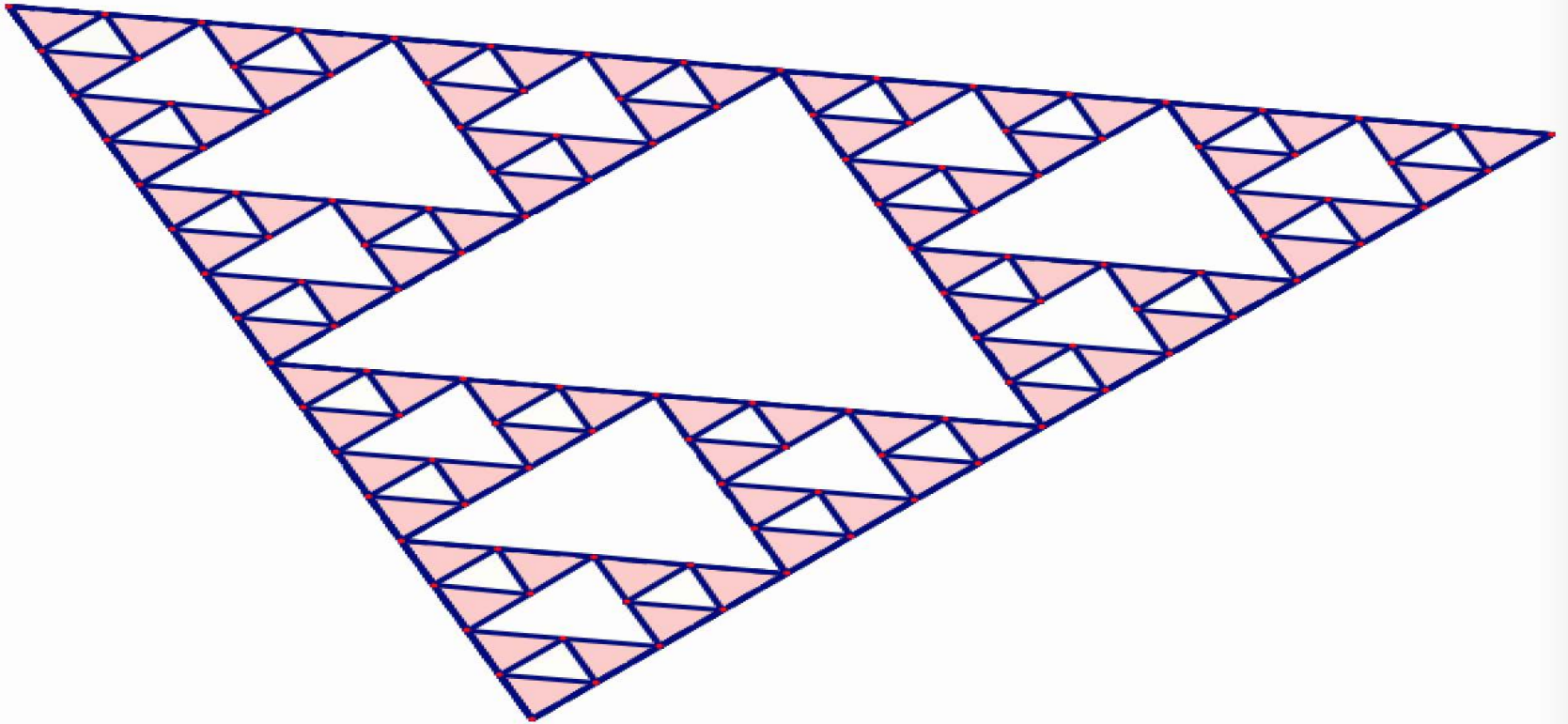
Same Figure - With Center Triangles Not Shaded



Process Will Work With Any Shaped Triangle



Another Example With 4 Iterations



Part II - Method Two (Group Activity)

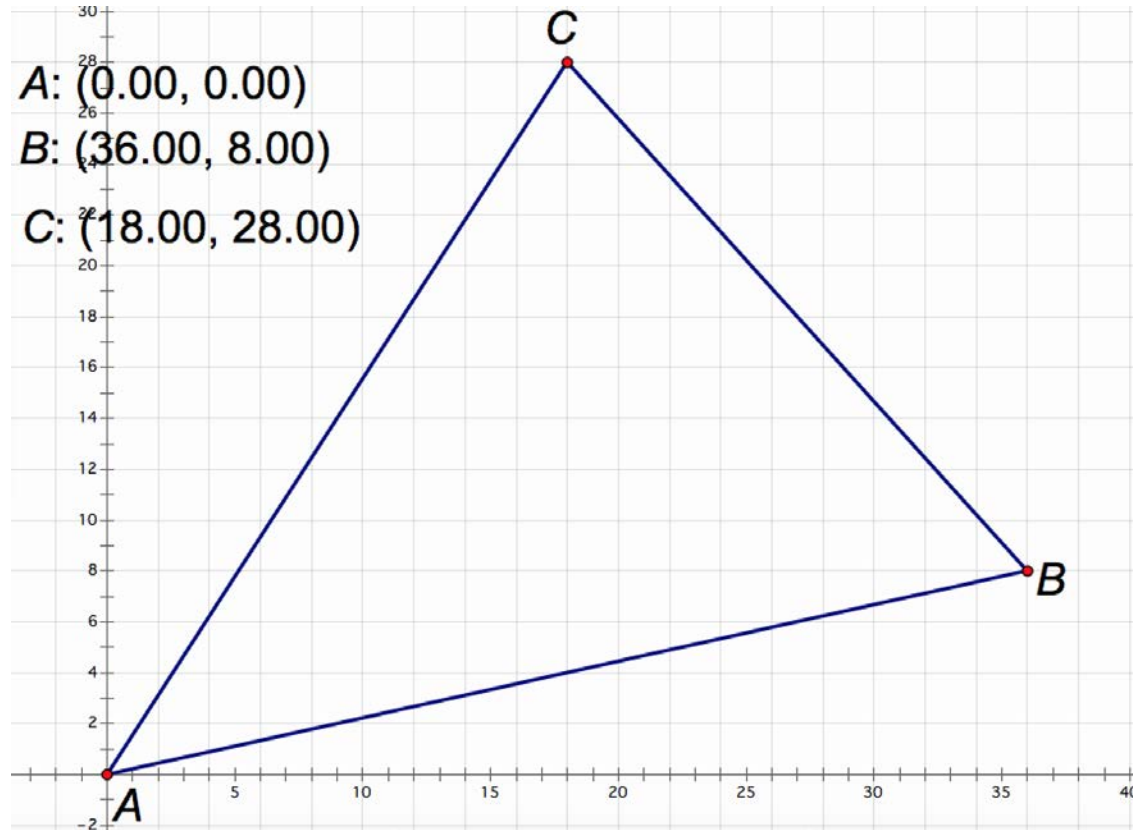
Supplies: graph paper, ruler, pencil, colored pencil or highlighter, die.

Optional- clear overhead transparency and appropriate marker.

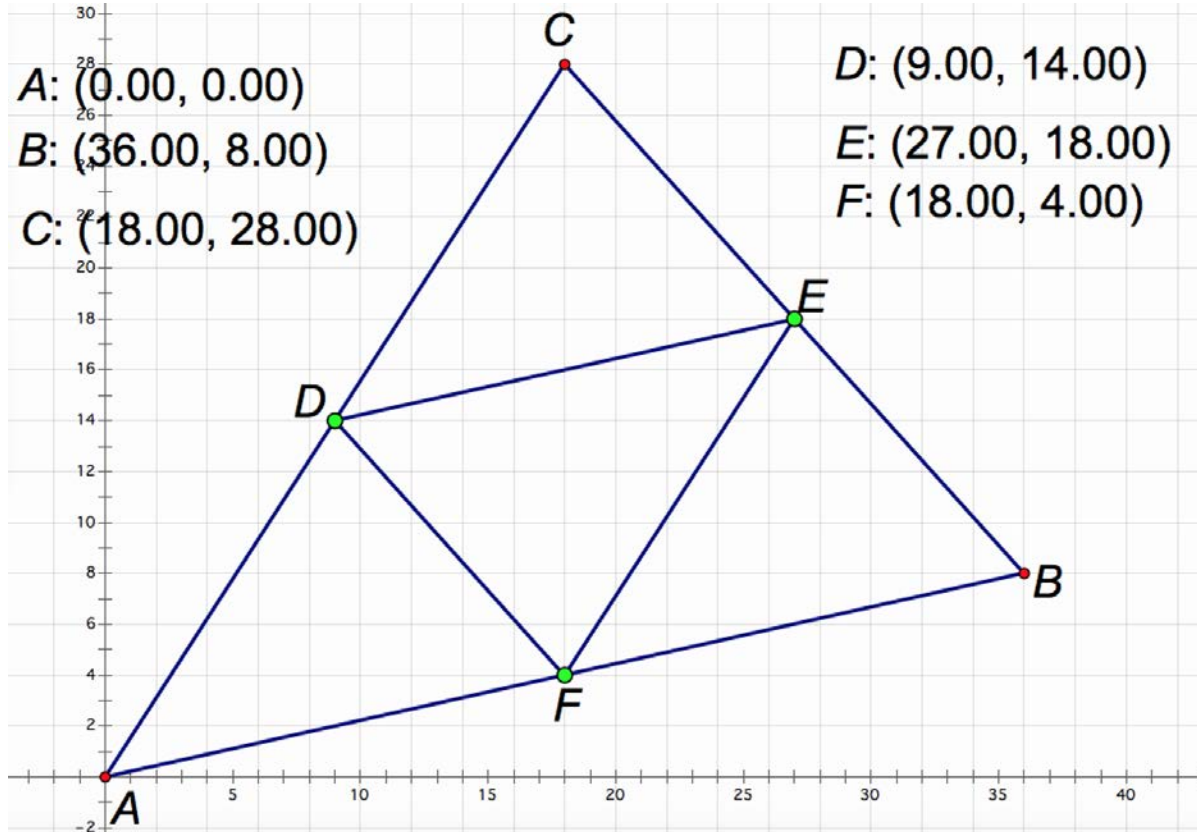
Each group makes one triangle.

1. Draw axes close to left and bottom side of the paper.
2. Pick three points to make a large triangle. It will be easier if one of the points is the origin and one of the points lies on one of the axes. Do not try to make a right or equilateral triangle. Label the points A, B, C.
3. Calculate the midpoints of each of the sides and graph the points.
4. Connect the midpoints.

Start - Draw Triangle on Coordinate Plane



Calculate & Connect Midpoints of Triangle Sides



Optional Add-on Activity

On a clear transparency, trace the original triangle you drew on the graph paper.

Using the technique from Part I, create the Sierpinski triangle on the transparency.
Do at least three or four iterations.

Set aside until after doing the midpoint calculations.

Challenge Time!

Pick any point P with coordinates (a, b) that lies anywhere inside the triangle.

Let each person in the group choose a different vertex.

Connect P with the chosen vertex. Calculate the midpoint of the new segment. Put the point on the segment. Color the point with the highlighter or colored pencil. *(Have each person use a different color, to help track the individual student's points.)*

Each person will be doing different calculations, but putting the points into the same graph.

For Your Next Points (each person works separately)

Roll the die. If you roll 1 or 2, choose vertex A. If you roll 3 or 4, choose vertex B. If you roll 5 or 6, choose vertex C. Connect the new midpoint you just found with the new chosen vertex.

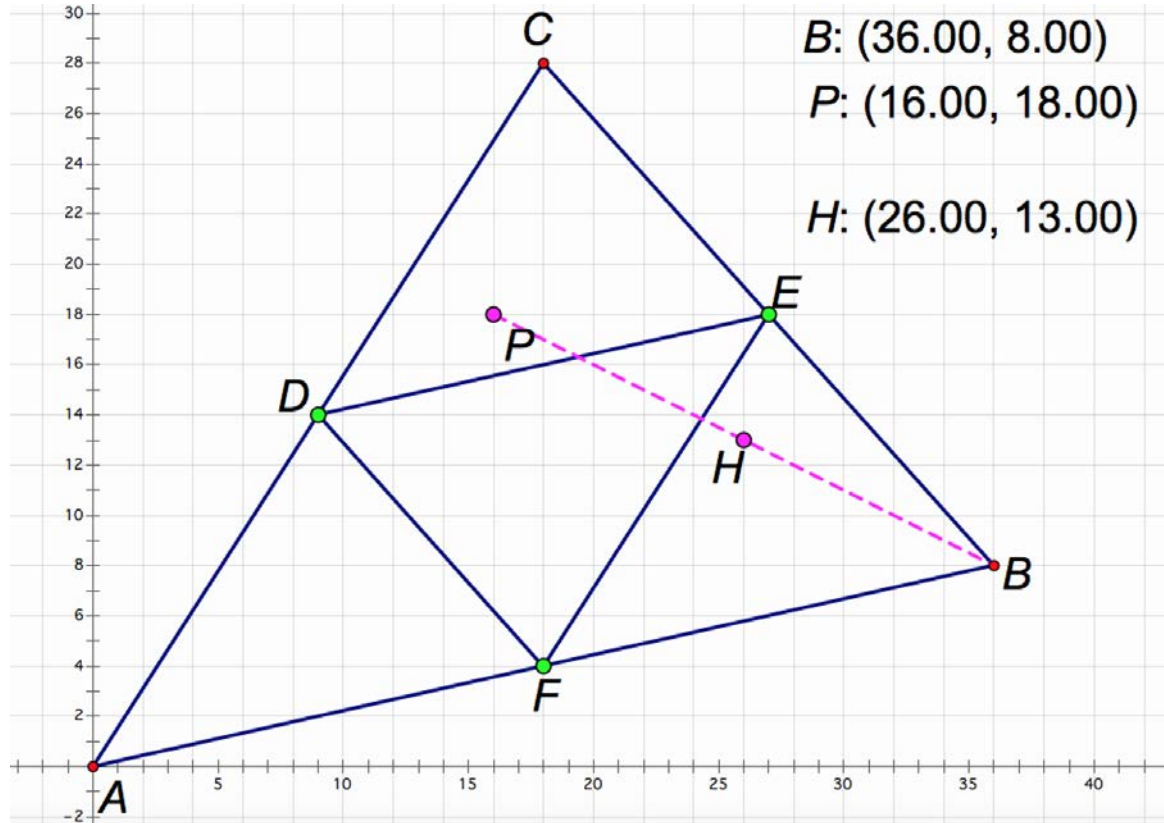
Calculate the midpoint of the new segment. Graph it and color it as before.

Repeat at least two more times.

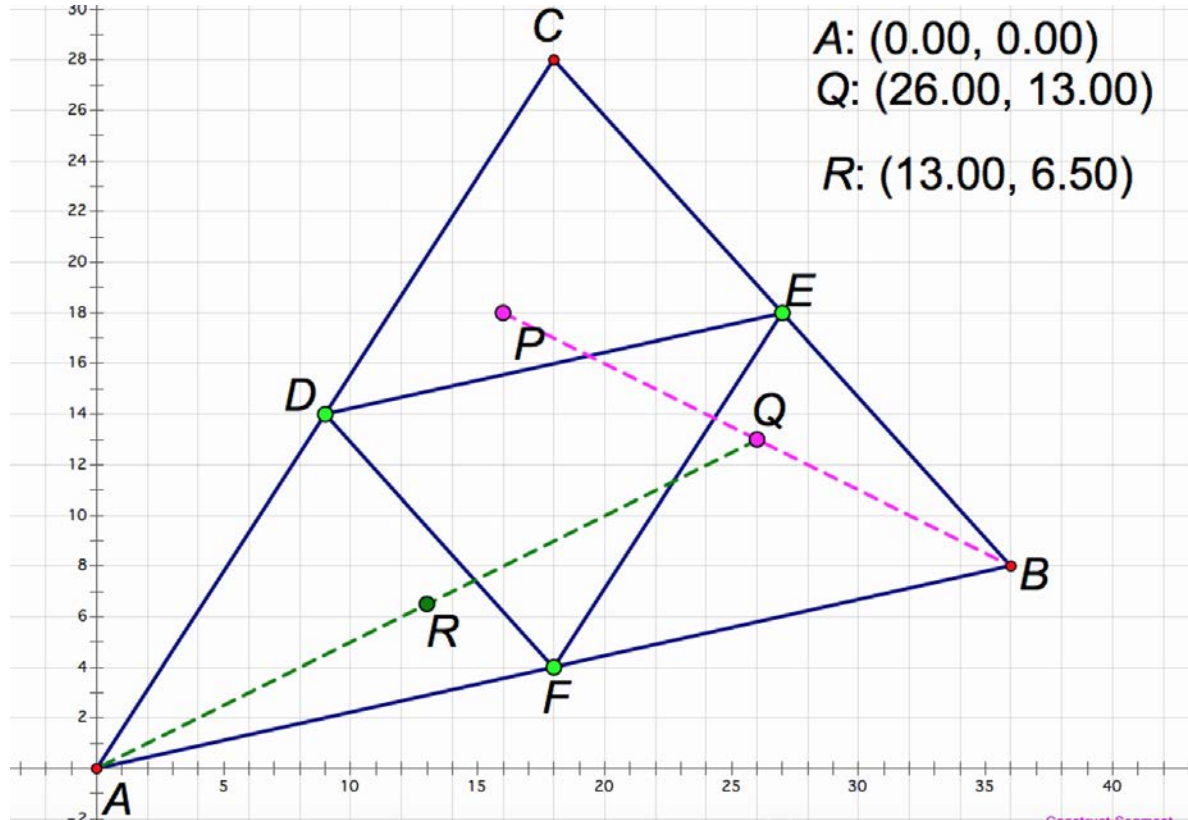
(If students are working independently, more points will be needed to give clearer results.)

Plot Point P (Anywhere) & Determine 1st Midpoint (H)

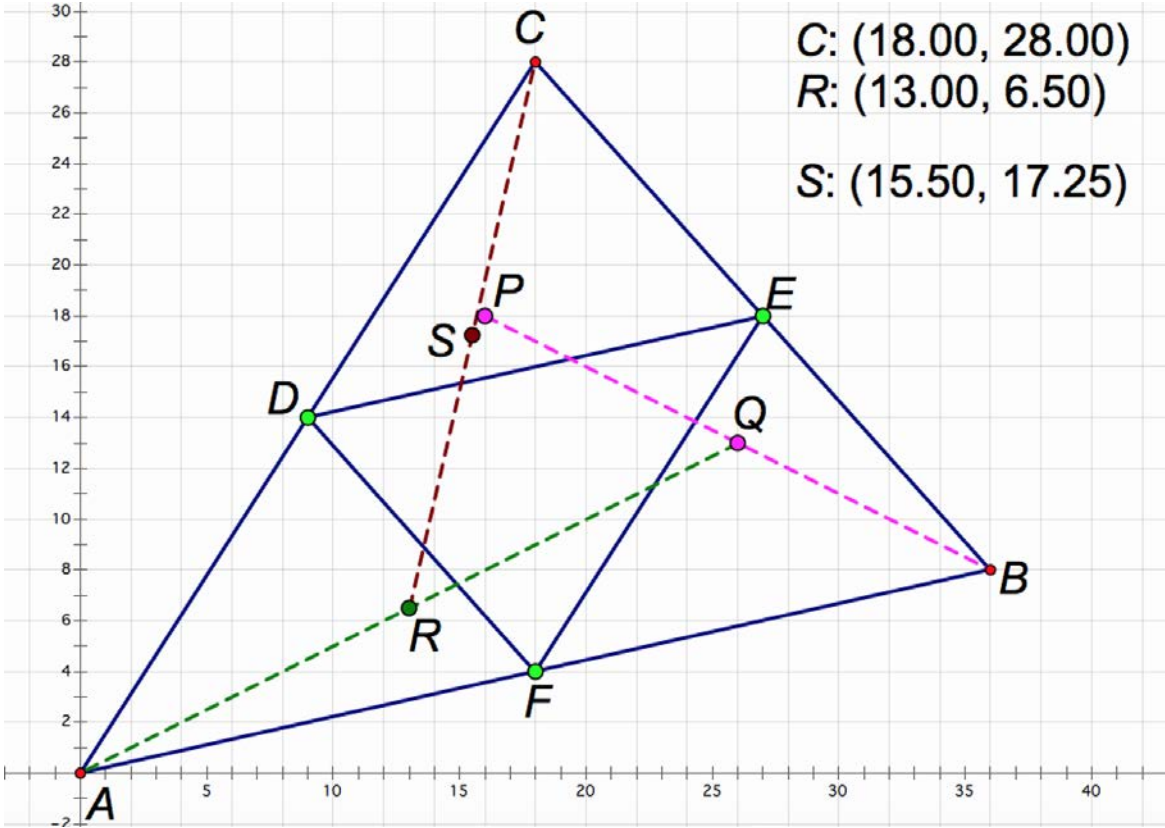
Find midpoint between P and any vertex (A,B, or C).



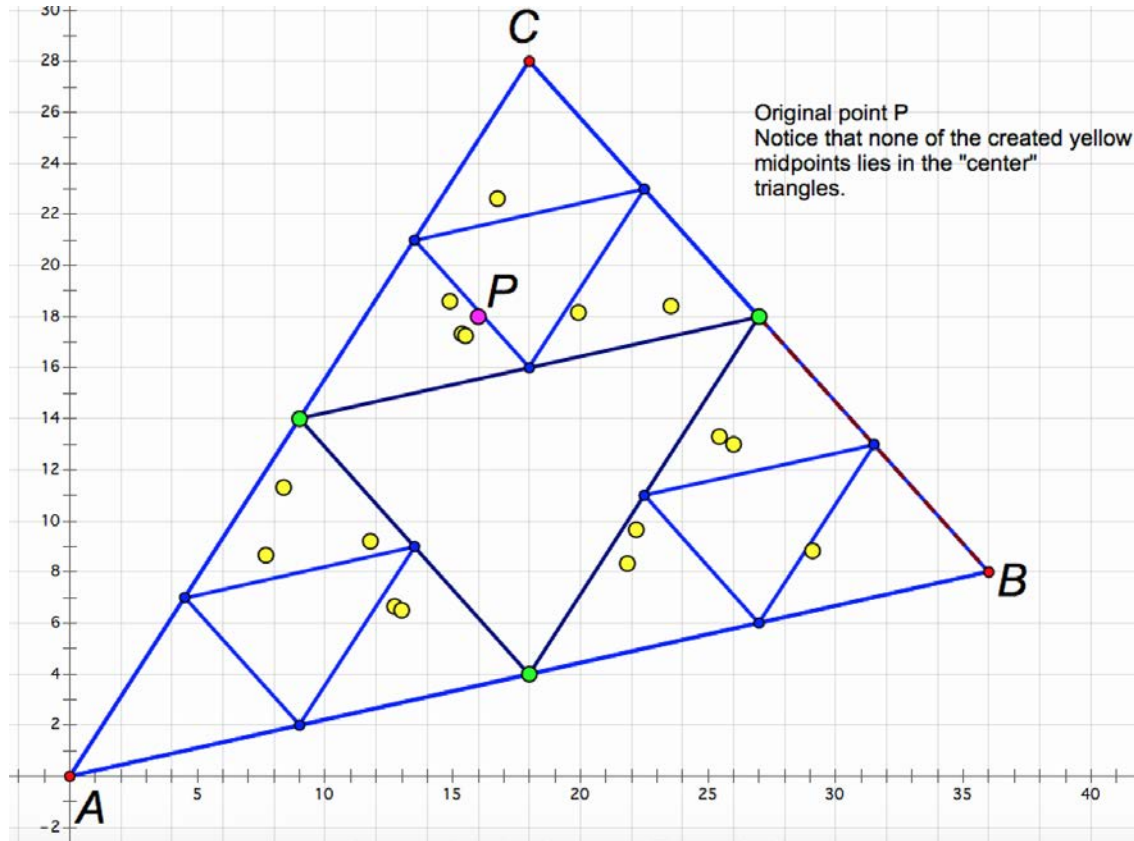
Determine 2nd Midpoint (R)



Determine 3rd Midpoint (S)



After 16 Iterations of Determining Midpoints



Results?

After each student has done at least three or four points and has graphed them all onto the same graph paper, study the graph.

What do you see forming?

How does it compare to Part I?

If you did the transparency, lay it over the graph.

What do you see?

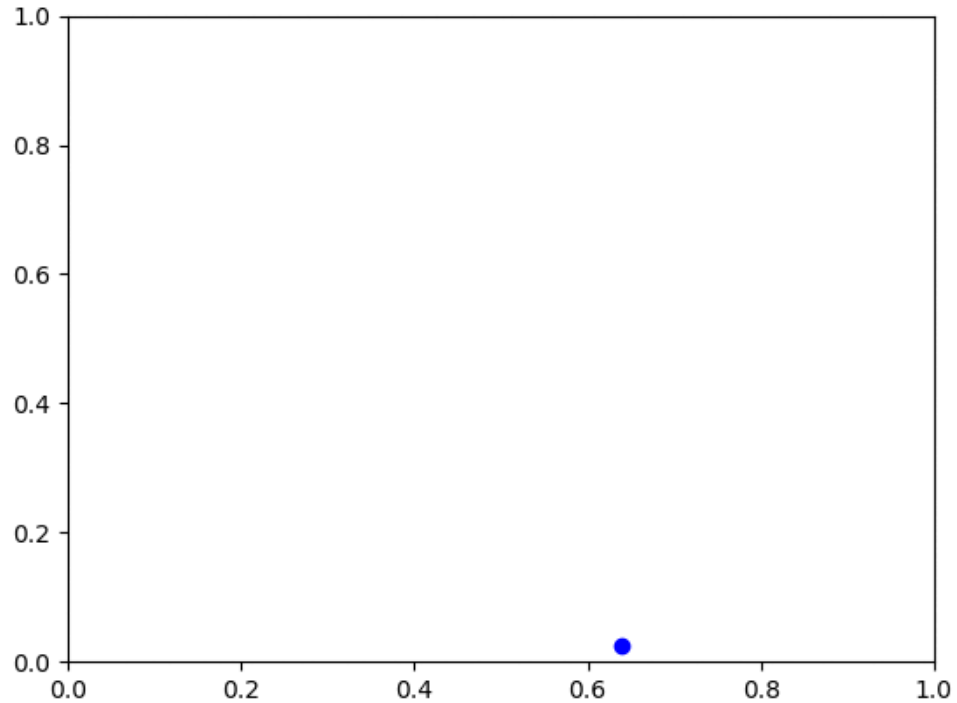
Part III - Now Try Using a Sierpinski Triangle Generator

<https://www.khanacademy.org/computer-programming/chaos-game/2777397046>

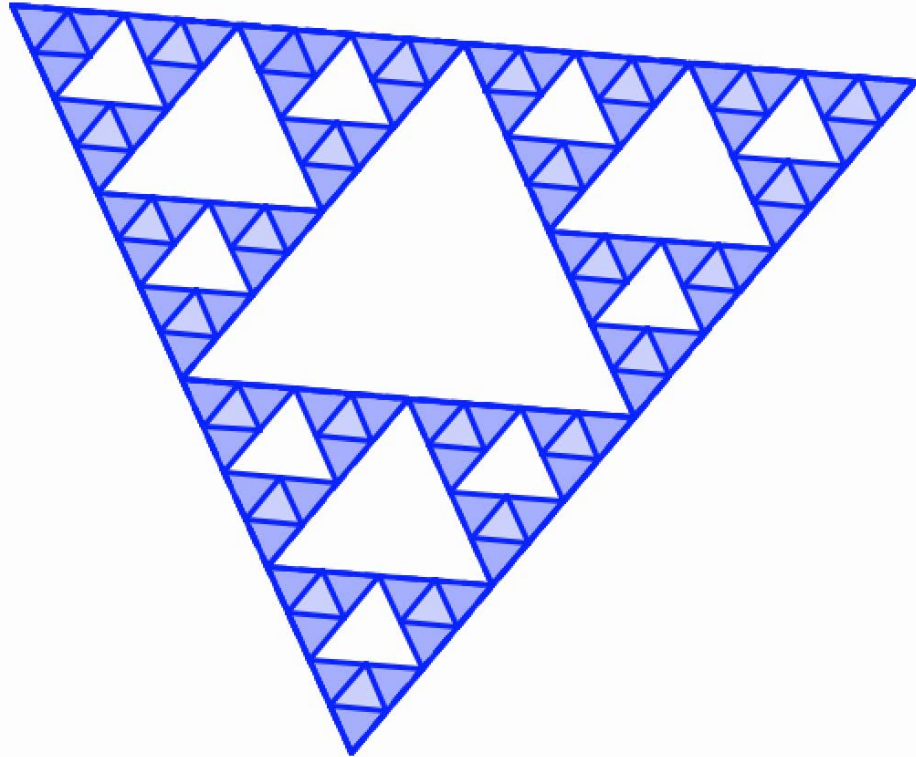
- Try changing the coordinates of the triangle
 - (x_1, y_1)
 - (x_2, y_2)
 - (x_3, y_3)
- Try changing the iterations
- Try changing the size of the dots- (line 24, var rad)

Another Sierpinski Triangle Generator

The computer lets us get more points than we did with paper and pencil.

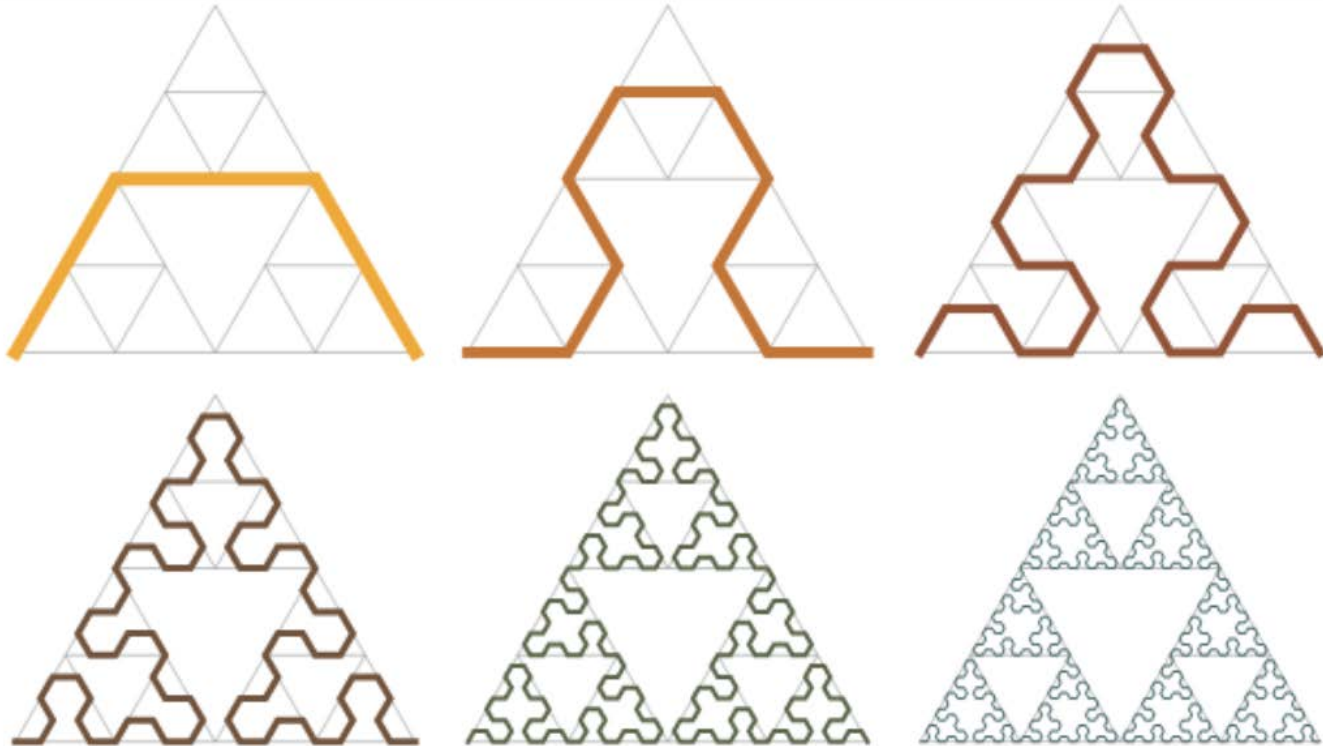


What else can we do with the Sierpinski Triangles and fractals?



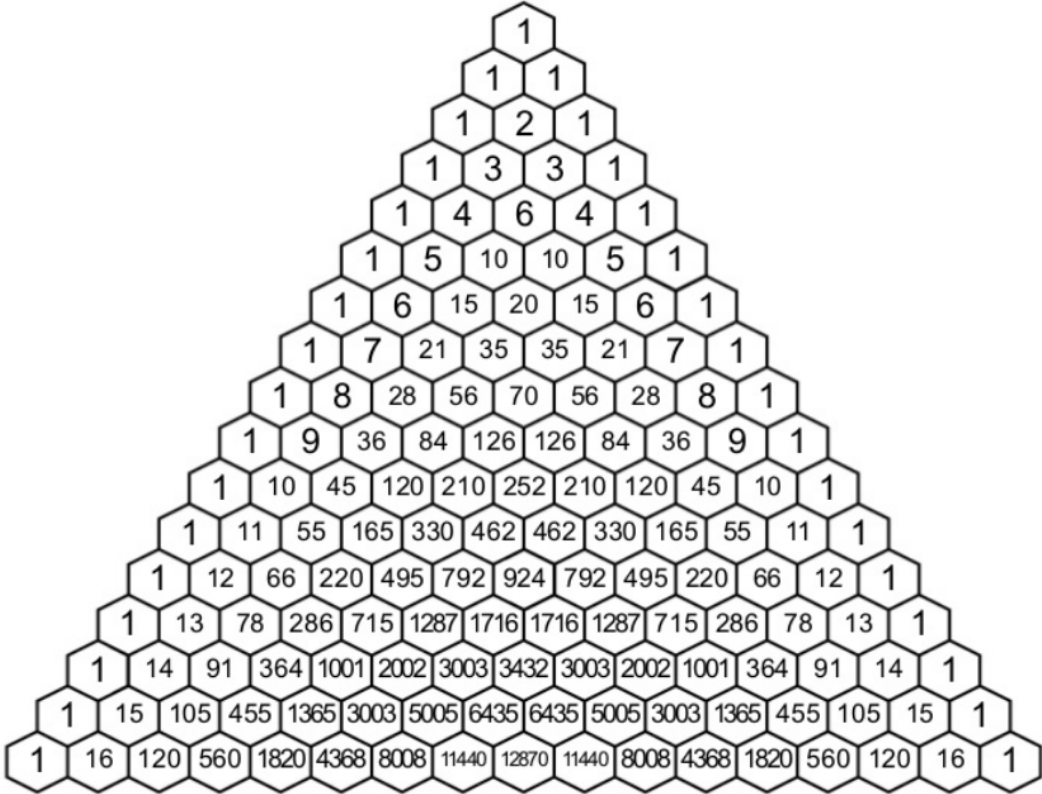
Another Way to Create a Sierpinski Triangle- Sierpinski Arrowhead Curve

Start with one line segment, then replace it by three segments which meet at 120 degree angles. The first and last segments are either parallel to the original segment or meet it at 60 degree angles.

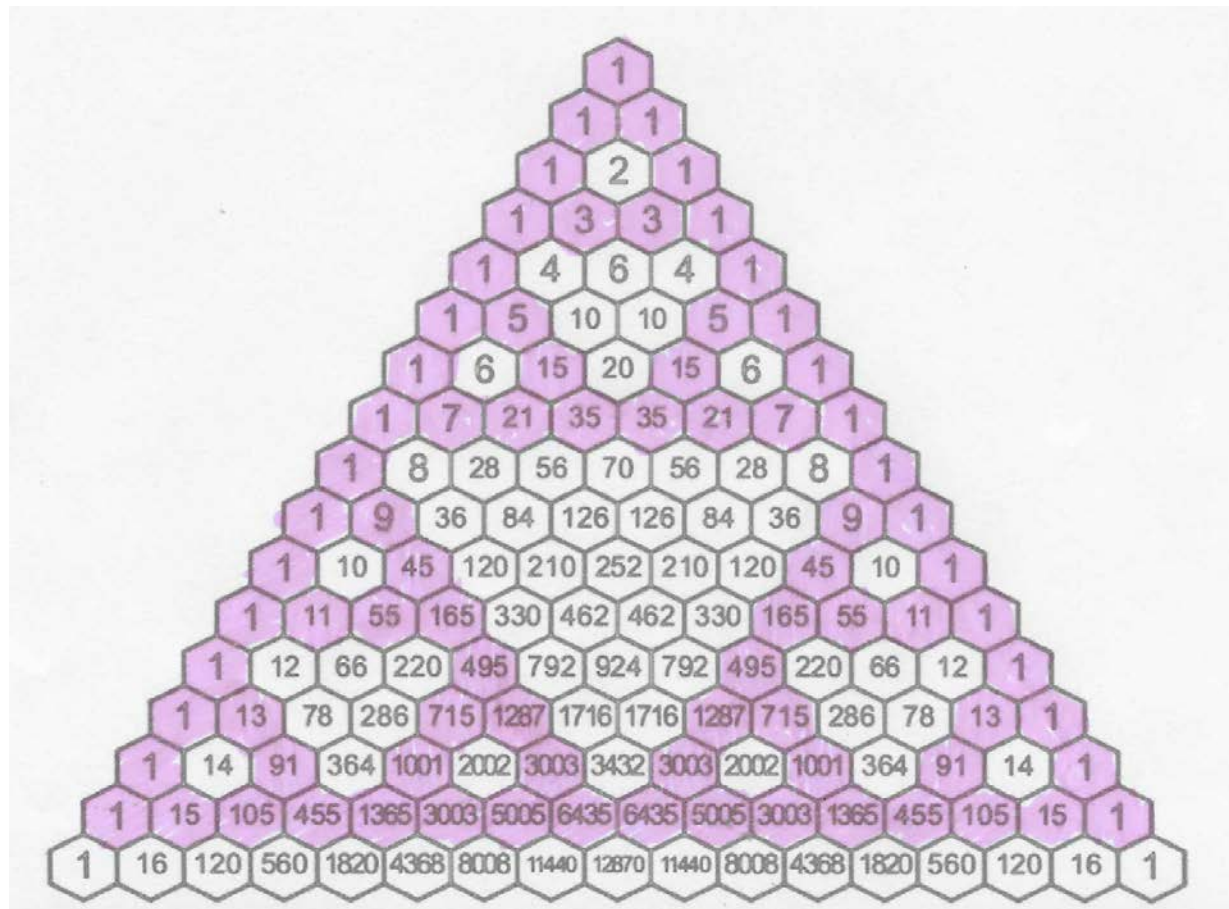


Sierpinski and Pascal

Color each odd number in Pascal's triangle and look at the result.



Sierpinski and Pascal



Geometric Connections

What geometric ideas that you have studied can be found in the Sierpinski Triangle?

Midline Theorem

Similar Triangles

Congruent Triangles

Transformations

Programming Extension

Start with

Midpoint Formula

$A(x_1, y_1)$
 $B(x_2, y_2)$
 $C(x_3, y_3)$
 $P(a, b)$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

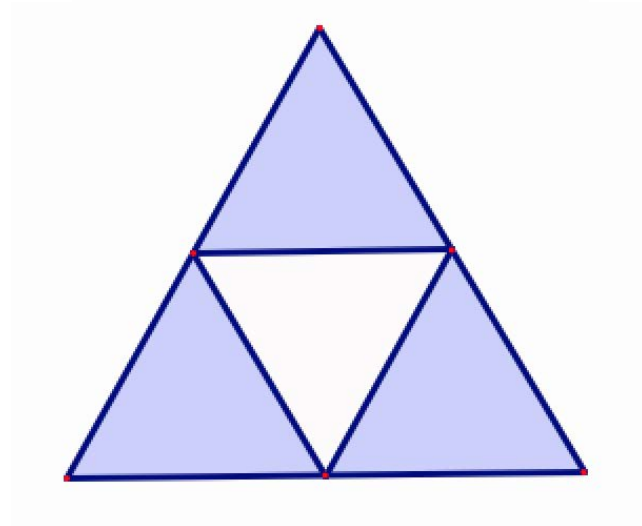
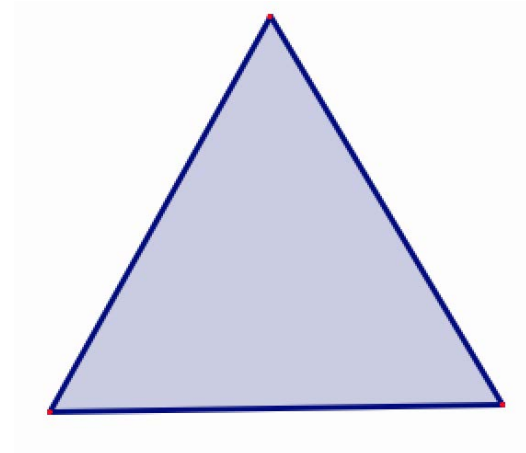
Subtraction Approach

Start with a solid triangle.

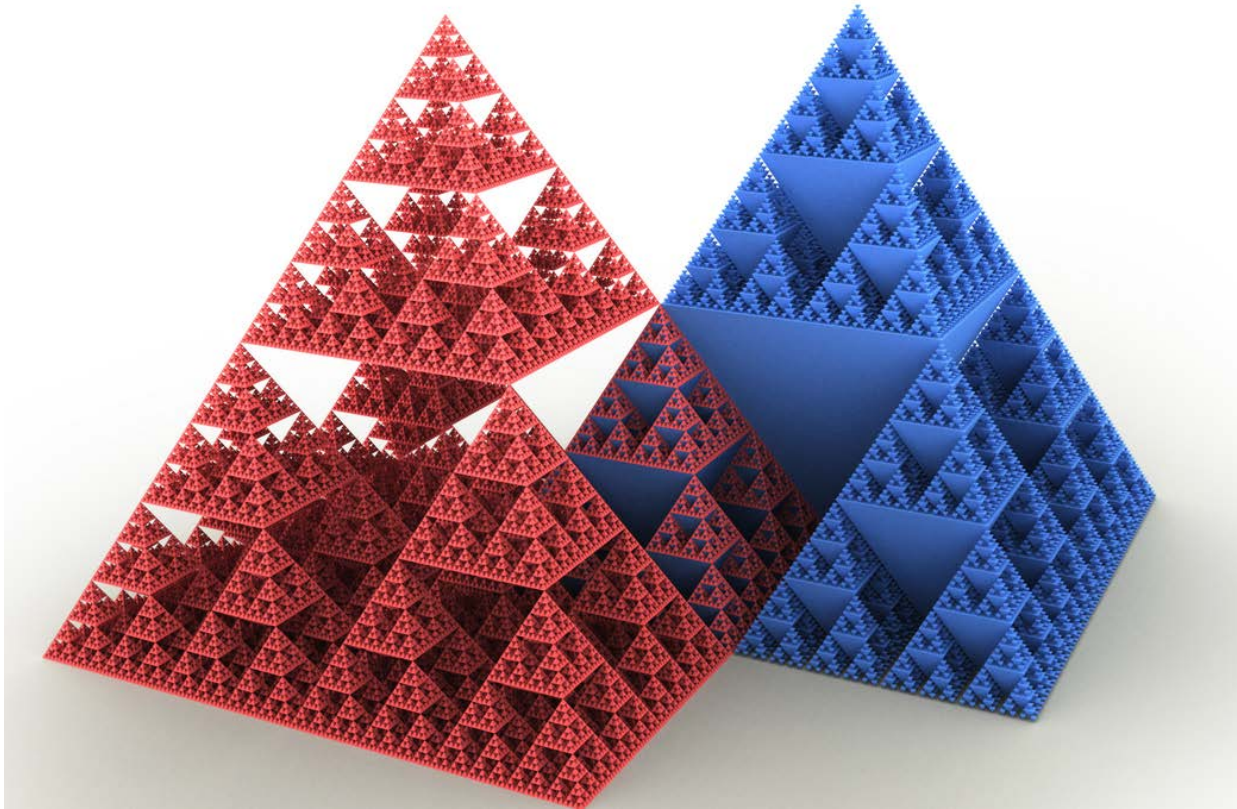
Divide into four congruent smaller triangles, using the midpoints of each side.

“Cut out” the center triangle.

Repeat for each of the remaining solid triangles.



Extension to Three Dimensions- Sierpinski Tetrahedron



Creative Challenge

Could you make a similar design if you started with a different polygon- square, hexagon, etc.?

What if you chose a different point (not the midpoint)?

(For example, a point one-third of the way from one vertex to another.)

Fractal Extension- Mandelbrot Set

Consider the values generated by complex number c and whole number z

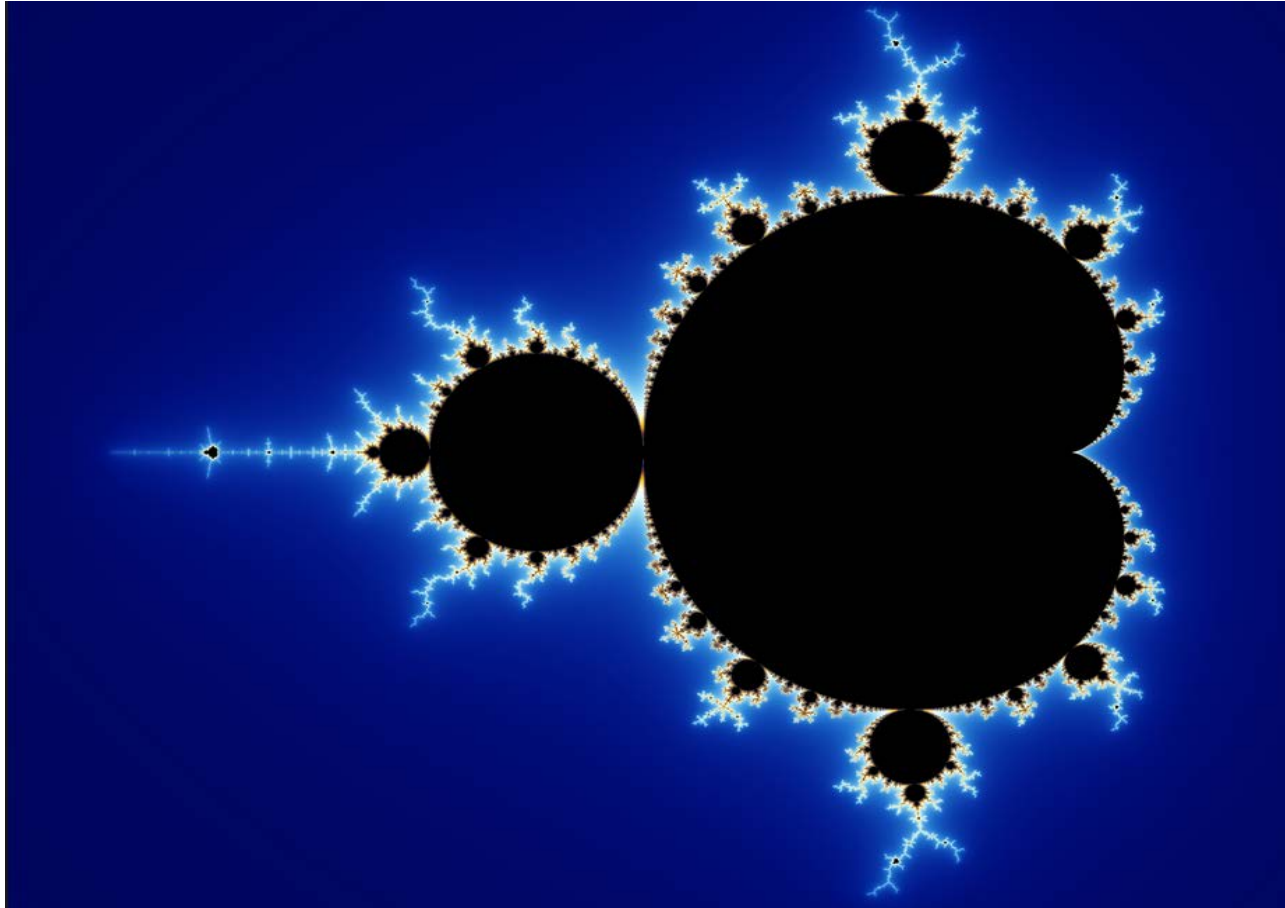
$$z_{n+1} = z_n^2 + c$$

Begin with $z_0 = 0$.

The number c is in the Mandelbrot set if the magnitude of z is bounded.

The black shape of the picture represents all complex numbers c where the absolute value of z is bounded by some number.

Mandelbrot Set Image



Math Standards

G-CO:1 parallel lines

G-CO:6 congruent triangles through transformations

G-CO:7 congruent triangle theorems

G-CO:9 theorems about lines, parallelism, angles

G-CO:10 theorems about triangles